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THE THEORY OF MOMENTLESS SHELLS OF REVOLUTION

By V. Z. Vlasov

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§1. Basic equations of the theory of momentless shells of revolution. By a momentless shell, we mean a thin-walled three-dimensional structural shape, whose natural, unloaded state is described by any given surface, and which does not show resistance to bending deformation at any point. Only tangential (normal and shearing) stresses acting parallel to the middle surface can arise out of the internal forces in a momentless shell. The intensity of these stresses is distributed uniformly across the thickness of the shell.

Let the shell be defined by a surface of revolution whose equation is $r = r(z)$, where z is the coordinate along the axis of revolution, and r is the radius of a parallel (Fig. 1).

In this case, the equations of equilibrium and extensional deformation of the middle surface of the shell of revolution have the form

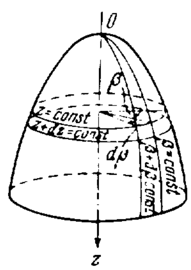


Fig. 1.

$$\begin{aligned} &\frac{\partial}{\partial z}(rN_1) - r'N_2 + A\frac{\partial S}{\partial \beta} = -Arp_\xi \\ &A\frac{\partial N_2}{\partial \beta} + \frac{\partial}{\partial z}(rS) + r'S = -Arp_\eta \\ &-\frac{rr''}{A^2}N_1 + N_2 = -Arp_\zeta. \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\frac{1}{A}\frac{\partial u}{\partial z} - \frac{r''}{A^3}w = \varepsilon_1 \\ &\frac{1}{r}\frac{\partial v}{\partial \beta} + \frac{r'}{Ar}u + \frac{1}{Ar}w = \varepsilon_2, \quad \frac{1}{r}\frac{\partial u}{\partial \beta} + \frac{r}{A}\frac{\partial}{\partial z}\left(\frac{v}{r}\right) = \omega \end{aligned} \tag{1.2}$$

Here N_1, N_2, and S are components of the tensor describing the stress state in the momentless shell (Fig. 2). $A = \sqrt{1 + (r')^2}$; A is a coefficient in the first quadratic form of the surface. p_ξ, p_η, and p_ζ are components of the surface load vector.

*Translated from: Izv. AN SSSR, OTN, no. 5, 1955, pp. 55-84.

ϵ_1, ϵ_2, and ω are the components of membrane strain in the shell; u, v, and w are components of the total displacement vector of a point n the middle surface of the shell (Fig. 3).

For any assigned surface loading, the internal stresses in the shell of revolution are determined by the static equations (1.1). The positive sense of these stresses is indicated on Fig. 2.

For any assigned membrane strains, the displacements of a point in the middle surface of the shell are determined by the geometric equations (1.2). The positive sense of these displacements is shown on Fig. 3.

§ 2. Homogeneous equations for momentless shells. Let us examine the problem of the equilibrium and bending of a momentless shell under the assumption that the right members of equations (1.1) and (1.2) are equal to zero. Setting $p_\xi = p_\eta = p_\zeta = 0$, we have the homogeneous, static stress equations describing the equilibrium of a momentless shell in the absence of a surface load:

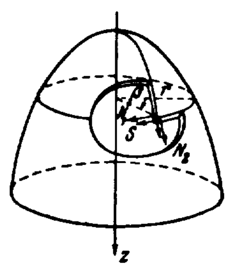


Fig. 2.

$$\begin{gathered} \frac{\partial}{\partial z}(rN_1) - r'N_2 + A\frac{\partial S}{\partial \beta} = 0 \\ A\frac{\partial N_2}{\partial \beta} + \frac{\partial}{\partial z}(rS) + r'S = 0 \\ -\frac{rr''}{A^2}N_1 + N_2 = 0 \end{gathered} \qquad (2.1)$$

Analogously, considering the tangential strains in the shell, ϵ_1, ϵ_2, and ω to be equal to zero, we have a system of three linear homogeneous equations for the displacements u, v, and w, of points in the middle surface of the shell.

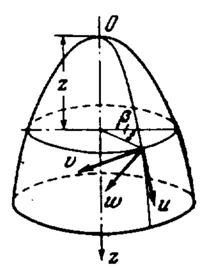


Fig. 3.

$$\begin{gathered} \frac{\partial u}{\partial z} - \frac{r''}{A^2}w = 0 \\ \frac{1}{r}\frac{\partial v}{\partial \beta} + \frac{r'}{Ar}u + \frac{1}{Ar}w = 0 \\ \frac{1}{r}\frac{\partial u}{\partial \beta} + \frac{r}{A}\frac{\partial}{\partial z}\left(\frac{v}{r}\right) = 0 \end{gathered} \qquad (2.2)$$

Bending deformations of an inextensible surface are, in fact, described by these equations. In terms of the flexural strains κ_1, κ_2, and τ, an infinitesimally small bending deformation of the surface of revolution can be described also by the equations

$$\frac{\partial}{\partial z}(r\varkappa_2) - r'\varkappa_1 + A\frac{\partial\tau}{\partial\beta} = 0$$
$$A\frac{\partial\varkappa_1}{\partial\beta} + \frac{\partial}{\partial z}(r\tau) + r'\tau = 0 \qquad (2.3)$$
$$-\frac{rr''}{A^2}\varkappa_2 + \varkappa_1 = 0$$

obtained on the basis of the static-geometric analogy from the equations of equilibrium (2.1) by replacing N_2 by κ_1, N_1 by κ_2, and S by τ.

Equations (2.3) are identical with equations (2.1).

Each of the individual systems of differential equations (2.1), (2.2), and (2.3), which pertain to three different problems for arbitrary shells of revolution, can be reduced to a system of two differential equations in two unknown functions. Introducing stress functions, displacement functions and strain functions into the investigation, and making use of the methods of the static-geometric analogy as set forth in the monograph [1]*, we can reduce each of the three systems of equations in such a manner that this system will have one and the same form for all three of the problems. We represent this system in the following form:

$$\frac{\partial V_i}{\partial z} + \frac{1}{r^2}\frac{\partial U_i}{\partial\beta} = 0, \qquad \frac{\partial V_i}{\partial\beta} + \frac{1}{rr''}\frac{\partial U_i}{\partial z} = 0 \qquad (i = 1,2,3) \qquad (2.4)$$

Here $U_i = U_i(z, \beta)$ and $V_i = V_i(z, \beta)$ are the new unknown functions. The index i assumes the values 1, 2, or 3, according to the number of the problem, described by equations (2.1), (2.2), or (2.3). Let us assume that equations (2.4) together with the functions determined by it with i = 1 are equivalent to equations (2.1); with i = 2, to equations (2.2); and with i = 3, to equations (2.3).

In accordance with this chosen system of enumeration, we have these formulas for the unknown functions of the original equations (2.1), (2.2), and (2.3):

$$N_1 = \frac{A}{r}V_1, \qquad N_2 = \frac{r''}{A}V_1, \qquad S = \frac{1}{r^2}U_1 \qquad (2.5)$$
$$u = \frac{1}{A}U_3, \qquad v = rV_3, \qquad w = \frac{A}{r''}\frac{\partial U_3}{\partial z} - \frac{r'}{A}U_3 \qquad (2.6)$$
$$\varkappa_2 = \frac{A}{r}V_2, \qquad \varkappa_1 = \frac{r''}{A}V_2, \qquad \tau = \frac{1}{r^2}U_2 \qquad (2.7)$$

Here, as before,

$$A = \sqrt{1 + r'^2}, \quad r = r(z), \qquad r' = \frac{dr}{dz}, \qquad r'' = \frac{d^2r}{dz^2}$$

With formulas (2.5), equations (2.4) are equivalent to equations (2.1); with formulas (2.6), to equations (2.2); and with formulas (2.7), to equations (2.3).

* The [1] appears to be a typographical error for [8] – Translators note.

Equations (2.4) are likewise reduced to a single differential equation of second order by introducing, for each of the three problems, one of the auxiliary functions F_i (z, β) according to the formulas

$$U_i = -r^2 \frac{\partial}{\partial z}\left(\frac{F_i}{r}\right), \qquad V_i = \frac{1}{r}\frac{\partial F_i}{\partial \beta} \tag{2.8}$$

This equation, associated with formulas (2.8), has the form

$$\frac{\partial^2 F_i}{\partial z^2} - \frac{r''}{r}\left(F_i + \frac{\partial^2 F_i}{\partial \beta^2}\right) = 0 \tag{2.9}$$

Here $F_i = F_i(z, \beta)$ is the fundamental function sought for. Depending on the character of the problem, it is either a stress function for the momentless shell, or a strain function for the inextensible surface, or, finally, a displacement function.

Formulas (2.5), (2.6), and (2.7), determining the stresses, displacements, and bending deformations take on the following form:

$$N_1 = \frac{A}{r}\frac{\partial F_1}{\partial \beta}, \qquad N_2 = \frac{r''}{A}\frac{\partial F_1}{\partial \beta}, \qquad S = -r\frac{\partial F_1}{\partial z} \tag{2.10}$$

$$u = -\frac{r^2}{A}\frac{\partial F_2}{\partial z}, \qquad v = r\frac{\partial F_2}{\partial \beta}, \qquad w = -\frac{A}{r''}\frac{\partial}{\partial z}\left(r^2\frac{\partial F_2}{\partial z}\right) + \frac{r^2 r'}{A}\frac{\partial F_2}{\partial z} \tag{2.11}$$

$$\varkappa_2 = \frac{A}{r}\frac{\partial F_3}{\partial \beta}, \qquad \varkappa_1 = \frac{r''}{A}\frac{\partial F_3}{\partial \beta}, \qquad \tau = -r\frac{\partial F_3}{\partial z} \tag{2.12}$$

Here F_1 is a stress function, F_2 is a displacement function, and F_3 is a strain function. With this modification, each of the problems, pertaining to the equilibrium and bending of a momentless inextensible shell, is described by a single differential equation (2.9) and its associated appropriate static or geometric boundary conditions.

§ 3. Elliptic and spherical shells. In our previous papers [1, 2], it was shown that if a shell is formed along a quadric surface having positive Gaussian curvature, i.e., along a surface which is convex everywhere (ellipsoid, sphere in a special case, paraboloid, hyperboloid of two sheets), then the equations of equilibrium of the momentless shell, and consequently also the equations of bending of the inextensible surface, can by means of the appropriate transformation be reduced to the Cauchy-Riemann equations of the theory of functions of a complex variable, or, equivalently, to the single harmonic equation

$$\frac{\partial^2 \varphi_i}{\partial \alpha^2} + \frac{\partial^2 \varphi_i}{\partial \beta^2} = 0 \tag{3.1}$$

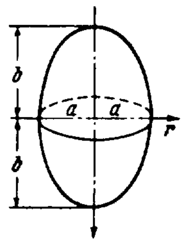


Fig. 4.

Here β is the angular coordinate, and α is determined by the formula (Fig. 4)

$$\alpha = \ln \sqrt{\frac{b+z}{b-z}}$$

The index i can, as before, take on all three of the numerical values 1, 2, and 3, corresponding to the three problems here examined simultaneously. Substituting for elliptical shells into the formulas (2.5), (2.6), and (2.7)

$$r = \frac{a}{\operatorname{ch}\alpha}, \qquad r' = -\frac{a}{b}\operatorname{sh}\alpha$$

$$r'' = \frac{d^2r}{dz^2} = -\frac{a\operatorname{ch}^3\alpha}{b^2}, \qquad A = \sqrt{1+(r')^2} = \frac{1}{b}\sqrt{b^2 + a^2\operatorname{sh}^2\alpha}$$

and identifying the functions $U_i = \frac{a^2}{b}\frac{\partial\varphi_i}{\partial\beta}$, $V_i = \frac{\partial\varphi_i}{\partial\alpha}$ $(i = 1, 2, 3)$, we obtain

$$N_1 = \frac{1}{ab}\operatorname{ch}\alpha\sqrt{b^2 + a^2\operatorname{sh}^2\alpha}\,\frac{\partial\varphi_1}{\partial\alpha}, \qquad S = \frac{1}{b}\operatorname{ch}^2\alpha\,\frac{\partial\varphi_1}{\partial\beta} \tag{3.2}$$
$$N_2 = -\frac{a}{b}\frac{\operatorname{ch}^3\alpha}{\sqrt{b^2 + a^2\operatorname{sh}^2\alpha}}\frac{\partial\varphi}{\partial\alpha},$$

$$u = \frac{a^2}{\sqrt{b^2 + a^2\operatorname{sh}^2\alpha}}\frac{\partial\varphi_2}{\partial\beta}, \qquad v = \frac{a}{\operatorname{ch}\alpha}\frac{\partial\varphi_2}{\partial\alpha} \tag{3.3}$$
$$w = -\frac{a}{b}\frac{\sqrt{b^2 + a^2\operatorname{sh}^2\alpha}}{\operatorname{ch}\alpha}\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta} + \frac{a^3}{b}\frac{\operatorname{sh}\alpha}{\sqrt{b^2 + a^2\operatorname{sh}^2\alpha}}\frac{\partial\varphi_2}{\partial\beta}$$

$$\varkappa_2 = \frac{1}{ab}\operatorname{ch}\alpha\sqrt{b^2 + a^2\operatorname{sh}^2\alpha}\,\frac{\partial\varphi_3}{\partial\alpha}, \qquad \tau = \frac{1}{b}\operatorname{ch}^2\alpha\,\frac{\partial\varphi_3}{\partial\beta} \tag{3.4}$$
$$\varkappa_1 = -\frac{a}{b}\frac{\operatorname{ch}^3\alpha}{\sqrt{b^2 + a^2\operatorname{sh}^2\alpha}}\frac{\partial\varphi_3}{\partial\alpha},$$

By means of formulas (3.2), the internal tangential specifications of the stress state in the momentless, elliptical shell, N_1, N_2, and S, are expressed in terms of the partial derivatives of the harmonic function φ_1. This function, which with the selected coordinates has the dimensions of force, is thus a stress function.

The formulas (3.3) determine, through partial derivatives of another function φ_2, all three components u, v, and w, of the total displacement vector of a point in the inextensible middle surface of an elliptical shell, arising as a consequence of the bending deformations of this surface. The function entering into these formulas can be called the harmonic displacement function of the elliptical shell.

Finally, all three components of the strain tensor in the case of infinitesimally small bending deformation of the elliptical surface, are determined by the last group of formulas (3.4). The formulas for these components have the same form

* In this paper 'th' ≡ 'tanh,' 'ch' ≡ 'cosh,' 'sh' ≡ 'sinh' – Publisher.

as the formulas (3.2). The function φ_3 is the fundamental harmonic flexural strain function of an elliptical surface of revolution.

Setting b = a in formulas (3.2, 3.3, 3.4), we obtain as a special case, the fundamental formulas for a spherical shell:

$$N_1 = \frac{1}{a}\,\mathrm{ch}^2\alpha\,\frac{\partial\varphi_1}{\partial\alpha}, \qquad S = \frac{1}{a}\,\mathrm{ch}^2\alpha\,\frac{\partial\varphi_1}{\partial\beta}, \qquad N_2 = -\frac{1}{a}\,\mathrm{ch}^2\alpha\,\frac{\partial\varphi_1}{\partial\alpha} \tag{3.5}$$

$$\varkappa_2 = \frac{1}{a}\,\mathrm{ch}^2\alpha\,\frac{\partial\varphi_3}{\partial\alpha}, \qquad \tau = \frac{1}{a}\,\mathrm{ch}^2\alpha\,\frac{\partial\varphi_3}{\partial\beta}, \qquad \varkappa_1 = -\frac{1}{a}\,\mathrm{ch}^2\alpha\,\frac{\partial\varphi_3}{\partial\alpha} \tag{3.6}$$

$$u = \frac{a}{\mathrm{ch}\,\alpha}\,\frac{\partial\varphi_2}{\partial\beta}, \qquad v = \frac{a}{\mathrm{ch}\,\alpha}\,\frac{\partial\varphi_2}{\partial\alpha}, \quad w = -a\,\frac{\partial^2\varphi_2}{\partial\alpha\,\partial\beta} + a\,\mathrm{th}\,\alpha\,\frac{\partial\varphi_2}{\partial\beta} \tag{3.7}$$

§ 4. <u>The hyperbolic shell.</u> All of the fundamental formulas for a hyperbolic shell can be obtained from the corresponding formulas for the elliptical shell examined above, by replacing the actual semi-axis of the ellipse, <u>b</u>, by the imaginary semi-axis of the hyperbola, bi, and α by αi. In place of the harmonic equations (3.1) for the fundamental functions φ_i (i = 1, 2, 3), we now have the wave equation:

$$\frac{\partial^2\varphi_i}{\partial\alpha^2} - \frac{\partial^2\varphi_i}{\partial\beta^2} = 0 \tag{4.1}$$

Substituting into formulas (2.5, 2.6, 2.7)

$$r = \frac{a}{\cos\alpha}, \qquad r' = \frac{a}{b}\sin\alpha, \qquad r'' = \frac{a\cos^3\alpha}{b^2}, \qquad A = \frac{1}{b}\sqrt{b^2 + a^2\sin^2\alpha}$$

and expressing the functions U_i and V_i in these formulas through the partial derivatives of the appropriate functions φ_i of the wave equation (4.1), we obtain

$$N_1 = \frac{1}{ab}\cos\alpha\sqrt{b^2 + a^2\sin^2\alpha}\,\frac{\partial\varphi_1}{\partial\alpha}, \qquad S = \frac{1}{b}\cos^2\alpha\,\frac{\partial\varphi_1}{\partial\beta}$$

$$N_2 = -\frac{a}{b}\,\frac{\cos^3\alpha}{\sqrt{b^2 + a^2\sin^2\alpha}}\,\frac{\partial\varphi_2}{\partial\alpha} \tag{4.2}$$

$$u = -\frac{a^2}{\sqrt{b^2 + a^2\sin^2\alpha}}\,\frac{\partial\varphi_2}{\partial\beta}, \qquad v = \frac{a}{\cos\alpha}\,\frac{\partial\varphi_2}{\partial\alpha}$$

$$w = -\frac{a}{b}\,\frac{\sqrt{b^2 + a^2\sin^2\alpha}}{\cos\alpha}\,\frac{\partial^2\varphi_2}{\partial\alpha\,\partial\beta} + \frac{a^3}{b}\,\frac{\sin\alpha}{\sqrt{b^2 + a^2\sin^2\alpha}}\,\frac{\partial\varphi_2}{\partial\beta} \tag{4.3}$$

$$\varkappa_2 = \frac{1}{ab}\cos\alpha\sqrt{b^2 + a^2\sin^2\alpha}\,\frac{\partial\varphi_3}{\partial\alpha}, \qquad \tau = \frac{1}{b}\cos^2\alpha\,\frac{\partial\varphi_3}{\partial\beta}$$

$$\varkappa_1 = -\frac{a}{b}\,\frac{\cos^3\alpha}{\sqrt{b^2 + a^2\sin^2\alpha}}\,\frac{\partial\varphi_3}{\partial\alpha} \tag{4.4}$$

By means of these equations, the unknown functions of the homogeneous equations (2.1), (2.2), and (2.3) for a hyperbolic shell of negative curvature, are expressed in terms of the partial derivatives of three functions φ_i (i = 1, 2, 3), each of which satisfies the wave equation (4.1).

§ 5. Boundary value problems in the theory of equilibrium and bending of shells of revolution. 1. Let us examine the problem of equilibrium and bending of elliptical and hyperbolic shells, limited in height by the planes $z = z_1$ and $z = z_2$ (Fig. 5). In this case, the height of the shell in the direction of the z axis is

$$h = z_1 + z_2 \tag{5.1}$$

At the edges of the shell, the independent variable α has the value

a) for elliptical shells:

$$\alpha_1 = \operatorname{arc\,sh} \frac{z_1}{\sqrt{b^2 - z_1^2}}, \qquad \alpha_2 = \operatorname{arc\,sh} \frac{z_2}{\sqrt{b^2 - z_2^2}} \tag{5.2}$$

b) for hyperbolic shells:

$$\alpha_1 = \arcsin \frac{z_1}{\sqrt{b^2 + z_1^2}}, \qquad \alpha_2 = \arcsin \frac{z_2}{\sqrt{b^2 + z_2^2}} \tag{5.3}$$

Let us seek a solution of equations (3.1) and (4.1) in the form of the simple trigonometric series:

$$\varphi_i(\alpha, \beta) = \sum f_{in}(\alpha) \cos n\beta, \qquad \Psi_i(\alpha, \beta) = \sum g_{in}(\alpha) \cos n\beta \tag{5.4}$$

We obtain the following formulas for the coefficients in these series:

$$f_{in}(\alpha) = A_{in} \operatorname{sh} n\alpha + B_{in} \operatorname{ch} n\alpha, \qquad g_{in}(\alpha) = C_{in} \sin n\alpha + D_{in} \cos n\alpha \tag{5.5}$$

Here A_{in}, B_{in}, C_{in}, and D_{in} are constants of integration. Substituting from (5.5) into (5.4), we obtain

$$\begin{aligned} \varphi_i(\alpha, \beta) &= \sum (A_{in} \operatorname{sh} n\alpha + B_{in} \operatorname{ch} n\alpha) \cos n\beta \\ \Psi_i(\alpha, \beta) &= \sum (C_{in} \sin n\alpha + D_{in} \cos n\alpha) \cos n\beta \end{aligned} \tag{5.6}$$

In this way, these formulas represent the general solution of the fundamental equations for elliptic and hyperbolic shells – of the harmonic equation (3.1) for elliptic shells, and of the wave equation (4.1) for hyperbolic shells.

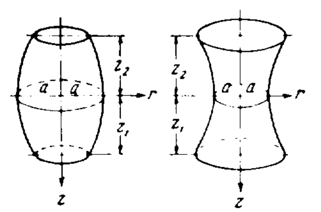


Fig. 5.

The index i, as before, can take on all three of the numerical values 1, 2, and 3, corresponding to the three different problems (one static and two geometric).

Calculating the partial derivatives with respect to α and β from the functions (5.6) just found, and then substituting these derivatives into the right members of the general equations (3.2), (3.3), (3.4), (4.2), (4.3), and (4.4), we obtain solutions in the form of simple series for all of the unknown static and geometric quantities pertaining to the two shells. All of these solutions are correctly determined up to the constants of integration of the corresponding problem. These constants must be found in each special case from the boundary conditions, which are given at the edges $\alpha = \alpha_1$ and $\alpha = \alpha_2$ to the extent of one condition throughout each edge for each of the three problems.

In the case of the purely static problem, we will assume that both the elliptic and hyperbolic shells are supported by flexible diaphragms at the edges $\alpha = \alpha_1$ and $\alpha = \alpha_2$, each acting only by forces lying in the planes of these diaphragms, and forming, together with the shell, a single three-dimensional closed system. With regard to external forces, we will assume that the shell is acted on by vertical edge loads, applied along the edges $\alpha = \alpha_1$, $\alpha = \alpha_2$, and given as a function of the angular coordinate β along each edge. With regard to the whole shell, this load must be a system of forces in static equilibrium. The action of the edge loads described here is expressed through the transmission to the shell, at points along the edges $\alpha = \alpha_1$ and $\alpha = \alpha_2$, of normal stresses N_1, directed along the tangent to an appropriate meridian, and which are themselves given functions of β. In accord with the requirements of the momentless theory, the radial components of the vertical load must be balanced by the plane diaphragms.

According to the first of formulas (3.2) and (4.2), the solution of the static problem described here reduces to the determination of stress functions φ_1 and Ψ_1, periodic in β, whose partial derivatives with respect to α must yield a given function of β along the parallels $\alpha = \alpha_1$ and $\alpha = \alpha_2$.

In the case of the purely geometric problem of the determination of the displacements, we shall assume that at the edges of the shell, $\alpha = \alpha_1$, $\alpha = \alpha_2$, there are only tangential displacements, given in magnitude as functions of β. In accord with the general formulas (3.3) and (4.3) for the displacements $\underline{v}$, the problem reduces also in this case to the determination of periodic functions φ_2 and Ψ_2, whose partial derivatives with respect to α must take on the values of a given function of β along the parallels $\alpha = \alpha_1$ and $\alpha = \alpha_2$.

In examining the other geometric problem, pertaining to the pure bending (without stretching) of the surface, we shall assume that this bending is specified by giving, as functions of β, the flexural deformations κ_2 along the edges of the shell $\alpha = \alpha_1$ and $\alpha = \alpha_2$. This boundary value problem has a complete analogy with the purely static one formulated above; and the solution of this problem reduces to finding the functions of infinitesimally small bending deformation, likewise periodic, which are designated as $\varphi_3(\alpha, \beta)$ and $\Psi_3(\alpha, \beta)$ for elliptic and hyperbolic shells respectively, and whose partial derivatives with respect to α at $\alpha = \alpha_1$ and $\alpha = \alpha_2$ must yield a given function of β in accordance with formulas (3.4) and (4.4) for κ_2.

Thus, for all three of the problems stated here, with the indicated boundary conditions, the auxiliary functions $\varphi_i(\alpha, \beta)$ of the harmonic equation (3.1) for elliptical shells and $\Psi_i(\alpha, \beta)$ of the wave equation (4.1) for hyperbolic shells, must be constructed such that the partial derivatives of these functions with respect to α at $\alpha = \alpha_1$ and $\alpha = \alpha_2$, reduce to given functions of β.

Representing these given functions for each of the three boundary value problems conforming to the general formulas (3.2-3.4, 4.2-4.4), in the form of trigonometric series in cos $n\beta$, and then imposing the boundary conditions involving the derivatives with respect to α of the fundamental unknown functions, we obtain systems of linear, algebraic equations for the coefficients of the trigonometric series (5.6). For the nth term of the pertinent series, these equations have the following form:

a) for elliptical shells

$$A_{ni}\,\mathrm{ch}\, n\alpha_1 + B_{ni}\,\mathrm{sh}\, n\alpha_1 = \frac{1}{n}\, p_{ni}$$
$$A_{ni}\,\mathrm{ch}\, n\alpha_2 + B_{ni}\,\mathrm{sh}\, n\alpha_2 = \frac{1}{n}\, q_{ni} \qquad (5.7)$$

b) for hyperbolic shells

$$C_{ni}\cos n\alpha_1 - D_{ni}\sin n\alpha_1 = \frac{1}{n}\, r_{in}, \qquad C_{ni}\cos n\alpha_2 - D_{ni}\sin n\alpha_2 = \frac{1}{n}\, s_{ni} \qquad (5.8)$$

As the number of the term in the pertinent trigonometric series, the index n can here take on any integral value; the index i designates to which one of the above-described separate problems equations (3.1) and (4.1) refer.

With i = 1, equations (5.7) and (5.8) will refer to the purely static problem of the equilibrium of momentless shells having, at the edges $\alpha = \alpha_1$ and $\alpha = \alpha_2$, diaphragms in the form of elastic membranes, and sustaining given normal stresses N_i at the edges $\alpha = \alpha_1$ and $\alpha = \alpha_2$.

With i = 2, we refer equations (5.7) and (5.8) to the purely geometric problem of infinitesimally small bending of shells with tangential displacements v given at the edges $\alpha = \alpha_1$ and $\alpha = \alpha_2$. Finally, with i = 3, these equations pertain to the infinitesimally small flexural strains of shells with flexural strains given at the edges $\alpha = \alpha_1$ and $\alpha = \alpha_2$.

For each index n (n = 1, 2, 3, ...), and for each of the three problems indicated above, the right members of equations (5.7) and (5.8) will be known quantities, proportional to the coefficients of the trigonometric series for the pertinent function which is given at the edges $\alpha = \alpha_1$ and $\alpha = \alpha_2$.

Solving equations (5.7) and (5.8), we obtain:

a) for elliptical shells

$$A_{ni} = \frac{1}{n}\frac{p_{ni}\,\mathrm{sh}\,n\alpha_2 - q_{ni}\,\mathrm{sh}\,n\alpha_1}{\mathrm{sh}\,n(\alpha_1+\alpha_2)}, \qquad B_{ni} = \frac{1}{n}\frac{q_{ni}\,\mathrm{ch}\,n\alpha_1 - p_{ni}\,\mathrm{ch}\,n\alpha_2}{\mathrm{sh}\,n(\alpha_1+\alpha_2)} \tag{5.9}$$

b) for hyperbolic shells

$$C_{ni} = \frac{1}{n}\frac{s_{ni}\sin n\alpha_1 - r_{ni}\sin n\alpha_2}{\sin n(\alpha_1-\alpha_2)}, \qquad D_{ni} = \frac{1}{n}\frac{s_{ni}\cos n\alpha_1 - r_{ni}\cos n\alpha_2}{\sin n(\alpha_1-\alpha_2)} \tag{5.10}$$

With given quantities p_{ni}, q_{ni}, r_{ni}, s_{ni} (n = 1, 2, 3 ..., i = 1, 2, 3), these formulas determine, generally speaking, all the coefficients of the trigonometric series (5.6), and consequently also the sought-for functions $\varphi_i(\alpha,\beta)$ and $\psi_i(\alpha,\beta)$ for all three different problems of elliptic and hyperbolic shells. However, a more detailed analysis of formulas (5.6) shows that hyperbolic shells of negative Gaussian curvature, both in regard to the momentless stress state and also in regard to their infinitesimally small bending, differ in principle from elliptical shells, which are shells of positive curvature. It is clear in formulas (5.9) that the determinant of the system (5.7), sinh $n(\alpha_1 - \alpha_2)$, with $\alpha_1 \neq \alpha_2$ cannot become zero for any of the values n = 1, 2, 3, ... With given finite p_{ni} and q_{ni}, the coefficients A_{ni} and B_{ni} thus take on completely definite finite values. Hence, it follows that with the system of boundary conditions indicated above, the harmonic function $\varphi_i(\alpha,\beta)$ admits a completely determined, and, moreover, unique solution, both in the case of the purely static problem, and also in the two other cases of purely geometric problems. In the case of the homogeneous boundary value problems, i.e., in the absence of normal stresses, tangential displacements, and flexural strains at all points along the edges of the shell, we will have the trivial zero solution for all three harmonic functions. As applied to the static problem, this means that if the elliptic shell, supported by plane membranes at the edges $\alpha = \alpha_1$ and $\alpha = \alpha_2$, carries no load whatsoever, then the internal stresses in this shell are everywhere equal to zero. A momentless, inextensible elliptical shell in the presence of plane shear* diaphragms at its edges is a three-dimensional,

* In the terminology of some American authors, a "shear diaphragm" is taken to represent an edge member whose rigidity within its own plane is infinite, but zero with respect to displacement normal to that plane – Translator's note.

statically determinate system. Static indeterminacy of such a shell can occur as a consequence of an excessive number of conditions pertaining to its edges. This indeterminacy can be resolved by examining the extensional strains of an elastic shell, which depend linearly on the internal tangential stresses.

As applied to the geometric problem, an elliptical inextensible shell whose edges, $\alpha = \alpha_1$, $\alpha = \alpha_2$, in the presence of shear diaphragms, cannot undergo any deformations in the planes of these diaphragms, is also, like a closed, inextensible sphere, a three-dimensional rigid surface. Infinitesimally small bending deformations of such a surface are impossible. These bending deformations are possible only if an edge of the shell does not have any constraint.

Analyzing formulas (5.6) pertaining to the hyperbolic shell of negative curvature, we see that the determinant of the system (5.8) can become zero for certain values of the argument $n(\alpha_1 - \alpha_2)$. These values are determined from the equation

$$n(\alpha_1 - \alpha_2) = m\pi \tag{5.11}$$

in which m and n can take on any integral values.

With given loads, displacements, and flexural strains at the edges $\alpha = \alpha_1$, $\alpha = \alpha_2$ of a hyperbolic shell, the coefficients C_{ni} and D_{ni} of the functions $\Psi_i(\alpha, \beta)$ of the wave equation can take on infinitely large values. In the absence of the indicated boundary values, formulas (5.6), together with (5.9) furnish an indeterminate solution for the functions Ψ_i of the wave equations. As applied to the static problem, this means that a hyperbolic shell of the Shukhov tower design, having "shear diaphragms" in the form of flexible but inextensible membranes at the $\alpha = \alpha_1$, $\alpha = \alpha_2$, in contrast to the elliptic shell, permits solutions for the internal stresses which are not only zero, but also different from zero, in the absence of any external load. And indeed, this also constitutes the static criterion of the infinitesimal geometric variability of momentless shells of negative curvature, first expressed in reference [6] and in monograph [8].

The purely geometric method, based on the notion of the static-geometric analogy, and proposed in the previously referred – to monograph [1]*, is in complete agreement with this criterion.

We also obtain direct confirmation of this method from the solution presented here. In fact, referring the wave function Ψ_i, determined with coefficients (5.10)

*The [1] appears to be a typographical error for [8] – Translator's note.

of formula (5.6), to the purely geometric problem, and supposing (in the case of the homogeneous boundary value problem) the displacements v_i and the flexural strains κ_2 at $\alpha = \alpha_1$ and $\alpha = \alpha_2$ to be equal to zero, i.e., assuming that the hyperbolic shell, as well as the elliptical shell examined above, is constrained at the edges $\alpha = \alpha_1$ and $\alpha = \alpha_2$ by diaphragms which are rigid in their planes, we have indeterminate, non-zero solutions for this function. These solutions are obtained from formulas (5.8) with $r_{ni} = 0$, $s_{ni} = 9$, and with the fulfillment of equation (5.11). This analysis shows that a hyperbolic shell of negative curvature, examined as an inextensible surface in the presence of shear diaphragms at the edges of the shell (each diaphragm freely deforming only out of its plane), can admit infinitesimally small bending deformations, in contrast to the similar elliptical shell.

All of these bending deformations, and the possible non-zero stress states which correspond to them in the sense of the static-geometric analogy, are described correctly to within constant multipliers by particular, unstable solutions of the wave equation, being, in fact, fundamental solutions of the homogeneous boundary value problem for this equation, and having a mathematical analogy with the problem of natural longitudinal vibrations of an elastic rod. These unstable eigensolutions of the wave equation are associated with those terms of the series (5.6) for which, in accordance with (5.8),

$$\sin n(\alpha_1 - \alpha_2) = 0 \tag{5.12}$$

Hence, we have:

$$\alpha_1 - \alpha_2 = \frac{m\pi}{n} \tag{5.13}$$

Here m and n are arbitrary, mutually independent whole numbers (m, n = 1, 2, 3, ...). Formula (5.12) can also be written in the form

$$\sin(\alpha_1 - \alpha_2) = \sin\frac{m\pi}{n} \tag{5.14}$$

Expressing the quantities α_1 and α_2 in this equation in terms of the coordinates of the edges of the shell z_1 and z_2 by means of formulas (5.3), we obtain

$$\frac{(z_1 - z_2)\,b}{\sqrt{(b^2 + z_1^2)(b^2 + z_2^2)}} = \sin\frac{m\pi}{n} \tag{5.15}$$

If the relative coordinates $\zeta_1 = z_1/b$, $\zeta_2 = z_2/b$ are now introduced for the edges of the shell, then equation (5.15) becomes

$$\frac{\zeta_1 - \zeta_2}{\sqrt{(1 + \zeta_1^2)(1 + \zeta_2^2)}} = \sin\frac{m\pi}{n} \tag{5.16}$$

Those relative dimensions for which the wave equation of the appropriate static or geometric homogeneous problem will have non-zero solutions are determined by equation (5.16). These critical dimensions bounding the shell along the axis of revolution can be called the critical heights.

Giving different integral values to the indices in formula (5.16), we will have, for the critical heights of the hyperbolic shell, an innumerable multitude of values which together form a spectrum of eigen-values for this parameter of the homogeneous boundary value problem.

If the plane z = 0 is a plane of symmetry for the shell, then formula (5.16), with a shell height of 2h, goes over into the following:

$$\frac{2bh}{b^2 + h^2} = \sin\frac{m\pi}{n} \tag{5.17}$$

Hence, we find

$$h = b\frac{1 \pm \cos m\pi/n}{\sin m\pi/n} \tag{5.18}$$

Since the quantity $m/n \leq 1$, with integral values of m and n, is a rational number of the unit interval (proper fraction), then hence it follows that for hyperbolic shells of negative curvature, there exist infinitely many values of the critical height h. Setting, in particular, n = 1, we obtain h = ∞.

This result shows that for shells of finite height, the first term of the series (5.6) will always yield a definite and, moreover, unique solution — different from zero in the case of the non-homogeneous problem, and zero in the case of the homogeneous problem.

This means that a hyperbolic shell loaded on the ends by normal forces distributed according to the law cos β or sin β, and reducing to a single bending moment with regard to the whole cross section (higher order moments corresponding to the remaining terms of the series n = 2, 3, 4, ..., are equal to zero), is a rigid system, and the internal stresses in the cross section of this system are distributed according to the law of plane sections.

Setting n = 2m, m = 1, 2, 3, ... in formula (5.18), we obtain h = b.

Hence, it follows that if the half height of the shell is equal to the imaginary semi-axis of the hyperbola generating this shell, then all the even numbered terms in the trigonometric series (5.6) will yield particular, unstable solutions taking on infinitely large values when the corresponding right-hand terms of equation (5.8) are different from zero, and indeterminate solutions with the right-hand terms equal to zero.

With $n = 3m$ and $m = 1, 2, 3, \ldots$, formula (5.18) for the critical height $\underline{h}$ gives two values: $h = \sqrt{3}b$ and $h = \sqrt{3}b/3$

With these values, the terms of the series (5.6) with the numbers $\underline{n}$ multiples of 3 ($n = 3, 6, 9, \ldots$), will yield infinitely large values for the unknown quantity in the case of the non-homogeneous boundary value problem, and indeterminate values in the case of the homogeneous problem.

Setting $n = 4m$, $m = 1, 2, 3, \ldots$, in formula (5.18), we obtain two new values for $\underline{h}$: $h = (\sqrt{2} + 1)b$ and $h = (\sqrt{2} - 1)b$.

With these values of the height $\underline{h}$, the special terms of the trigonometric series (5.6) will be associated with terms whose indices are multiples of 4($n = 4, 8, 12, \ldots$).

All the rest of the infinitely many values for the critical height $\underline{h}$ can be obtained from formula (5.18) in similar fashion.

We note that to the nth term of the series (5.6) for the critical height $\underline{h}$ there corresponds not one, but $(n-1)$ values. Thus, for example, for the fourth term of the series (5.6), formula (5.18) gives three values for $\underline{h}$: $h = b$, $h = (\sqrt{2} + 1)b$, and $h = (\sqrt{2} - 1)b$.

The first of these values is obtained with $n = 2m$ and $m = 2$; the two others with $n = 4m$ and $m = 1$.

2. We now examine the homogeneous boundary value problem of equilibrium and bending of a closed symmetrical shell consisting of three shells: two end elliptical shells and one middle elliptic or hyperbolic shell (Fig. 6). We will assume that the geometric dimensions of the end elliptical shells are such that there are common tangents to the meridians at the junctions of these shells with the middle shell. In other words, we assume that the closed shell of revolution, in each of the two cases illustrated in Fig. 6, is obtained by revolution of an appropriate compound curve, not having corner points at the lines of contact.

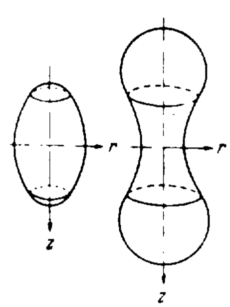


Fig. 6.

In the interest of simplicity of computation, without reducing the generality of the solution of the problem, we will examine only states of stress and bending of shells which are symmetrical with respect to the plane z = 0, as shown on Fig. 6. In this case, when formulas (5.6) are applied to the middle shell, all terms which are even functions of α drop out. The formulas assume the form:

$$\varphi_i = \sum A_{in} \operatorname{sh} n\alpha \cos n\beta, \qquad \Psi_i = \sum C_{in} \sin n\alpha \cos n\beta \tag{5.19}$$

Both of these formulas pertain to the middle shell of height 2h. The harmonic function for a middle elliptical shell is determined by the first of these formulas. The second formula gives the function satisfying the wave equation for a middle hyperbolic shell. The independent variable α in formulas (5.19) can range

from $\quad \alpha = -\operatorname{arc\,sh} \frac{h}{\sqrt{b^2 - h^2}} \quad$ to $\quad \alpha = \operatorname{arc\,sh} \frac{h}{\sqrt{b^2 - h^2}}$

in the case of the elliptical shell, and

from $\quad \alpha = -\arcsin \frac{h}{\sqrt{b^2 + h^2}} \quad$ to $\quad \alpha = \arcsin \frac{h}{\sqrt{b^2 + h^2}}$

in the case of the hyperbolic shell.

Applying the fundamental functions φ_i and Ψ_i, which are determined for the middle shells by formulas (5.19), to the static problem, we have these formulas for the stresses N_1 and S:

a) for an elliptical shell

$$\begin{aligned} N_1 &= \frac{1}{ab} \operatorname{ch} \alpha \sqrt{b^2 + a^2 \operatorname{sh}^2 \alpha} \sum nA_n \operatorname{ch} n\alpha \cos n\beta \\ S &= -\frac{1}{b} \cos^2 \alpha \sum nA_n \sin n\alpha \sin n\beta \end{aligned} \tag{5.20}$$

b) for a hyperbolic shell

$$\begin{aligned} N_1 &= \frac{1}{ab} \cos \alpha \sqrt{b^2 - a^2 \sin^2 \alpha} \sum nC_n \cos n\alpha \cos n\beta \\ S &= -\frac{1}{b} \cos^2 \alpha \sum nC_n \sin n\alpha \sin n\beta \end{aligned} \tag{5.20'}$$

Since formulas (5.20) pertain to a symmetrical state of equilibrium, then in the sequel we can limit our examination to either half of the shell, for example, the lower.

The harmonic function describing the stress state in the end elliptical shell must be constructed such that it is regular at all points of the ellipsoid, including the pole. This condition is satisfied by the following function:

$$\varphi' = \sum A_n' e^{-n\alpha'} \cos n\beta \tag{5.21}$$

Here A_n' is an arbitrary constant for the end elliptical shell, $\underline{n}$ is a positive integer ($n = 1, 2, \ldots$), α' is the independent variable for this shell, ranging in the extremes from $\alpha' = \alpha_1'$ to $\alpha' = \infty$. The cordinate value α_1' corresponds to the line of contact of the end shell with the middle one. The value $\alpha' = \infty$ corresponds to the lower pole of the lower end elliptical shell.

We have the following formulas for the internal stresses N_1' and S' of the lower end elliptical shell:

$$N_1' = -\frac{1}{a'b'}\operatorname{ch}\alpha'\sqrt{b'^2 + a'^2\operatorname{sh}\alpha'}\sum nA_n'e^{-n\alpha'}\cos n\beta \qquad (5.22)$$
$$S' = -\frac{1}{b'}\operatorname{ch}^2\alpha'\sum nA_n'e^{-n\alpha'}\sin n\beta$$

Here the primes indicate that the appropriate magnitudes pertain to an end elliptical shell. Since, according to the conditions of the problem, the closed shell of revolution has common tangents to the meridian curves along the parallels of contact of the end shells with the middle one, then the stress state for the entire compound shell must be such that the internal normal stresses N_1 and shearing stresses S satisfy continuity conditions at points along the lines of contact of the middle shell with the end ones, in the absence of any external load along these lines. We will have two mutually independent boundary conditions at the lines of contact:

$$N_1 - N_1' = 0, \qquad S - S' = 0 \qquad (5.23)$$

Here N_1 and S pertain to the middle shell, and are calculated from formulas (5.20) in the case of the elliptical shell and from formulas (5.20') in the case of the hyperbolic shell. In these formulas, the independent variable α is in both cases to be assigned the value α_1, corresponding to the lower-most parallel of the middle shell.

The stresses N_1' and S' in equations (5.23) are computed from formulas (5.22) for the lower end elliptical shell, with the coordinate value $\alpha' = \alpha_1'$, which determines the position of the line of contact of this shell with the middle one.

Applying the conditions (5.23) to a shell which has positive curvature everywhere as shown in Fig. 6, and taking into consideration the fact that the meridians of the joined shells have a common tangent at the point of contact, we obtain:

$$\frac{1}{a}\operatorname{ch}\alpha_1\operatorname{ch}n\alpha_1 A_n + \frac{1}{a'}\operatorname{ch}\alpha_1' e^{-n\alpha_1'}A_n' = p_n \qquad (5.24)$$
$$-\frac{1}{b}\operatorname{ch}^2\alpha_1\operatorname{sh}n\alpha_1 A_n + \frac{1}{b'}\operatorname{ch}^2\alpha_1' e^{-n\alpha_1'}A_n' = q_n$$

Here a and b are the semi-axes of the ellipse for the middle (central) shell; a' and b' are the semi-axes of the ellipse for the end shells; A_n and A_n' are unknown coefficients of the series (5.19) and (5.21); p_n and q_n are arbitrary terms which depend on the external load and are proportional to the nth term of the decomposition of this load into a trigonometric series.

In the case of two elliptical shells joined along a line of contact having a radius $r = r_1$, we have $\mathrm{ch}\,\alpha_1 = a/r_1$ and $\mathrm{ch}\,\alpha_1' = a'/r_1$. By virtue of these relations, equations (5.24) become:

$$\begin{aligned} A_n \,\mathrm{ch}\, n\alpha_1 + A_n' e^{-n\alpha_1'} &= r_1 p_n \\ -A_n \frac{a^2}{b} \,\mathrm{sh}\, n\alpha_1 + A_n' \frac{a'^2}{b'} e^{-n\alpha_1'} &= r_1^2 q_n \end{aligned} \qquad (5.25)$$

We obtain the following general formula for the determinant of these equations, pertaining to an arbitrary nth term of the trigonometric series:

$$\Delta_n = \left(\frac{a'^2}{b'} \,\mathrm{ch}\, n\alpha_1 + \frac{a^2}{b} \,\mathrm{sh}\, n\alpha_1\right) e^{-n\alpha_1'} \qquad (5.26)$$

In the case of the elliptical shell being examined, the quantities a, b, α_1', b', and n entering into these formulas can take on only positive values. Hence, it follows that the determinant Δ_n of equations (5.25) cannot reduce to zero for any value of the index n. Consequently, with assigned arbitrary (right-hand) terms, equations (5.25) furnish completely determinate values for A_n and A_n'. In the case of no load, all the coefficients A_n, A_n' of the trigonometric series (5.19) and (5.21) will vanish.

A thin-walled closed system consisting of shells of positive curvature (Fig. 6) is, like a closed spherical or elliptical shell, a three-dimensional, geometrically stable, and statically determinate system.

Making use of the method of the static-geometric analogy, and replacing N_1 and S by κ_2 and τ respectively in the preceding equations, we arrive at the conclusion that the closed compound shell having positive curvature everywhere, which is under consideration here, cannot have flexural strains in the absence of membrane strains. Such a shell, considered as an inextensible surface, is a rigid three-dimensional system.

Let us now examine a closed shell having negative curvature in its middle part, and consisting of one hyperbolic shell and two elliptical shells (Fig. 6). Let a and b be the semi-axes of the hyperbola for the middle shell, and let a_1' and b_1' be the semi-axes of the ellipse for an end shell. Writing out the static boundary conditions (5.23) with the help of the general formulas (5.20') and (5.22), recognizing also that the meridian segments of the complete shell have a common tangent at the point of contact of the hyperboloid with the ellipse, we obtain:

$$\begin{aligned} C_n \frac{1}{a} \cos\alpha_1 \cos n\alpha_1 + \frac{A_n'}{a'} \operatorname{ch}\alpha_1' e^{-n\alpha_1'} &= p_n \\ - C_n \frac{1}{b} \cos^2\alpha_1 \sin n\alpha_1 + \frac{A_n'}{b'} \operatorname{ch}^2\alpha_1' e^{-n\alpha_1'} &= q_n \end{aligned} \qquad (5.27)$$

Here C_n and A_n' are unknown coefficients of the series (5.19) and (5.21); p_n and q_n are arbitrary terms, which depend on the external load and reduce to zero in the case of no load. By substituting $\cos\alpha_1 = a/r_1$ and $\cosh\alpha_1' = a'/r_1'$ (where r_1 is the radius of the parallel at the junction) we obtain:

$$C_n \cos n\alpha_1 + A_n' e^{-n\alpha_1'} = r_1 p_n, \qquad - C_n \frac{a^2}{b} \sin n\alpha_1 + A_n \frac{a'^2}{b'} e^{-n\alpha_1'} = r_1^2 q_n \qquad (5.28)$$

We obtain the following formula for the determinant of the system of equations (5.28)

$$\Delta_n = e^{-n\alpha_1'} \left(\frac{a'^2}{b'} \cos n\alpha_1 + \frac{a^2}{b} \sin n\alpha_1 \right)$$

It follows from this formula that for a shell of the given type, there exist relative dimensions for which the determinant Δ_n has the value zero for positive integral values of the index n. For the values of $n\alpha_1$ which cause the determinant Δ_n to vanish, we obtain the equations

$$\operatorname{tg} n\alpha_1 = -\frac{a'^2 b}{a^2 b'} \quad \text{or} \quad \alpha_1 = -\frac{1}{n} \operatorname{arc\,tg} \frac{a'^2 b}{a^2 b'} \pm \frac{m}{n} \qquad (5.29)$$

Here m and n are any positive whole numbers.

With given dimensions a, b, a', b' of the compound shell, and with arbitrary integral values of the quantities m, n = 1, 2, 3, ... , the formula (5.29) gives infinitely many critical values of α_1.

Since the quantity α_1 is determined by the position of the line of contact of the middle shell with one of the end shells, according to the formula $\cos\alpha_1 = a/r_1$ which pertains to the hyperbolic shell, then infinitely many values of the parameter α_1 correspond to infinitely many such lines of contact, along which the equations of equilibrium, and consequently also the equations of bending of the compound shell examined here, will have unstable solutions. In contrast to a closed elliptical shell, a closed compound hyperbolic shell can permit infinitely many mutually independent shapes of infinitesimally small bending deformations. These bending deformations, like the possible shapes of equilibrium momentless states under no load which are analogous to them, are explained by the presence in the shell of a part with negative curvature.

3. If a shell is described by revolution of a segment of the curve

$$r = a\left(\cos\frac{z}{a} + \lambda\sin\frac{z}{a}\right) \tag{5.30}$$

not intersecting the axis of revolution Oz (a and λ are certain constant quantities), then equation (2.9) goes over into an equation with constant coefficients

$$\frac{\partial^2 F_i}{\partial z^2} + \frac{1}{a^2}\left(F_i + \frac{\partial^2 F_i}{\partial \beta^2}\right) = 0 \tag{5.31}$$

This is an equation of the elliptic type. Expressing the β-dependence of F(z,β) as an ordinary trigonometric series in which the coefficients are functions of _z_, we obtain, from (5.31)

$$F_i(\alpha, \beta) = \sum\left(A_{ni}\,\mathrm{sh}\,\frac{z\sqrt{n^2-1}}{a} + B_{ni}\,\mathrm{ch}\,\frac{z\sqrt{n^2-1}}{a}\right)\cos n\beta \tag{5.32}$$

Here _n_, for the shell of revolution can assume the value of any integer, and A_{ni} and B_{ni} are constants of integration in the nth term of the series (5.32). The index _i_ picks out one of the three problems (2.10), (2.11), or (2.12) to which the fundamental function F_i can refer.

Analysis of the problem described here shows that a shell of revolution with meridian given by equation (5.30) belongs to the class of shells of positive Gaussian curvature. These shells possess the same properties which were also examined in the above elliptical shells.

4. Let us now examine a shell of revolution with the meridian given by the equation

$$r = a\left(\mathrm{ch}\,\frac{z}{a} + \lambda\,\mathrm{sh}\,\frac{z}{a}\right) \tag{5.33}$$

With arbitrary values of the parameters _a_ and λ, these shells represent shells of negative curvature. In particular, with

$$r = a\,\mathrm{ch}\,\frac{z}{a} \tag{5.34}$$

we will have a shell whose middle surface is generated by a catenoid.

Using (5.34), equations (2.9) assume the form:

$$\frac{\partial^2 F_i}{\partial z^2} - \frac{1}{a^2}\left(F_i + \frac{\partial^2 F_i}{\partial \beta^2}\right) = 0$$

This equation is of the hyperbolic type. Upon representing it in the form of a single trigonometric series, the function $F_i(\alpha,\beta)$ assumes the form

$$F_i(\alpha, \beta) = \sum\left(C_{ni}\sin\frac{z}{a}\sqrt{n^2-1} + D_{ni}\cos\frac{z}{a}\sqrt{n^2-1}\right)\cos n\beta$$

Here, as before, $\underline{n}$ is an arbitrary whole number, and C_n and D_n are constants of integration.

By applying the above-outlined method of static and geometric analysis to a shell of the class (5.34), we arrive at the conclusion that, as shells of negative curvature, these shells are essentially different from shells of the elliptic type, and can admit particular, non-zero solutions both in the case of the homogeneous static problem, and also in the case of the homogeneous geometric problem.

5. Let us examine still another class of shells, generated by revolving, around the axis Oz, the algebraic curve

$$r = A(z - a)^{\mu} \tag{5.35}$$

Here A, $\underline{a}$, and μ are parameters of the curve. Since all of these parameters, including also the exponent μ, can take on any real values (integral, fractional, rational, irrational, positive, or negative), then equation (5.35) embraces an extremely wide class of shells of revolution. Differentiating (5.35), we find

$$r' = \frac{\mu r}{z - a}, \qquad r'' = \frac{\mu(\mu - 1) r}{(z - a)^2}$$

For the curvatures of the surface, we have the formulas

$$k_1 = -\frac{r''}{(1 + r'^2)^{3/2}} = -\frac{\mu(\mu - 1) r}{(z-a)^2 [1 + \mu^2 r^2 (z - a)^2]^{3/2}}$$

$$k_2 = \frac{1}{r(1 - r'^2)^{1/2}} = \frac{1}{r[1 + \mu^2 r^2/(z - a)^2]^{1/2}}$$

For the Gaussian curvature, we now obtain the formula

$$K = k_1 k_2 = -\frac{\mu(\mu - 1)}{(z - a)^2 [1 + \mu^2 r^2/(z - a)^2]}$$

It follows from this formula that the sign of the Gaussian curvature of the surface depends only on the exponent μ of the meridian curve (5.35).

As a consequence of this, we can divide surfaces, generated by revolving the curve (5.35), into three types:

a) Surfaces of positive curvature (paraboloids of different order)

$$K > 0 \qquad (0 < \mu < 1)$$

b) Surfaces of negative curvature (hyperboloids of different order)

$$K < 0 \qquad (1 < \mu < 0)$$

c) Surfaces of zero curvature

$$k = 0, \qquad \mu = 0, \qquad \mu = 1$$

For shells with surfaces of the class examined here, the fundamental equations (2.9) assume the form:

$$\frac{\partial^2 F}{\partial z^2} - \frac{\mu(\mu-1)}{z^2}\left(\frac{\partial^2 F}{\partial \beta^2} + F\right) = 0 \tag{5.36}$$

For shells of positive curvature, this equation will be of the elliptic type, and for shells of zero curvature, of the parabolic type.

Integrating equation (5.36) by expanding $F(z,\beta)$ as a trigonometric series in the angular coordinate β, with coefficients depending only on $\underline{z}$, we obtain,

$$F(z, \beta) = \sum\left(A_n z^{\frac{1+p_n}{2}} + B_n z^{\frac{1-p_n}{2}}\right)\cos n\beta \tag{5.37}$$

Here p_n is a characteristic number, depending on the index of the nth term of the series (5.37) and on the exponent of the meridian curve (5.35), and determined according to the formula:

$$p_n = \sqrt{1 - 4\mu(\mu - 1)(n^2 - 1)} \tag{5.38}$$

Since, for shells of revolution, $\underline{n}$ can assume integral values, then it follows from formula (5.38), that the quantities p_n will be real numbers for shells of positive curvature, and imaginary numbers for shells of negative curvature.

Determining all of the fundamental static and geometric quantities from the general formulas (2.10, 2.11, 2.12), and applying the method of initial functions proposed by the author in reference [1], we can, in this way, examine a whole series of new problems in the theory of equilibrium and bending of shells of the class examined here, and develop new rational engineering shapes for engineering construction.

Let us examine a shell of revolution of limited height. The constants of integration A_n and B_n of the general series (5.37) are determined by the boundary conditions, which must be given along the edges $z = z_1$ and $z = z_2$ to the extent of one static condition and one geometric condition all along the edges. In the static problem of the equilibrium of a momentless shell, the boundary conditions will pertain either to the normal stresses N_i or the shear stresses S.

In the geometric problem of the bending of an inextensible surface, the boundary conditions will pertain, in accordance with the static-geometric analogy, either to the flexural strains κ_2 or to the twisting strains τ.

Assuming, as before, that the shell has "shear diaphragms" in the form of flexible, inextensible plates at the transverse bounding edges $z = z_1$ and $z = z_2$, we will have a system of two homogeneous equations for the coefficients A_n and B_n in the case of the homogeneous boundary value problem.

$$A_n z_1^{\frac{-1+p_n}{2}} + B_n z_1^{\frac{-1-p_n}{2}} = 0, \qquad A_n z_2^{\frac{-1+p_n}{2}} + B_n z_2^{\frac{-1-p_n}{2}} = 0 \tag{5.39}$$

Equating the determinant of the system (5.39) to zero, we separate out that class of shells for which the equilibrium shape fails to show uniqueness. Such shells, considered as inextensible surfaces, are thin-walled, geometrically variable systems, which permit infinitesimally small bending deformations, even in the presence of transverse, inextensible diaphragms at the bounding parallels $z = z_1$ and $z = z_2$.

In this way, we obtain

$$\Delta_n = \frac{1}{\sqrt{z_1 z_2}}\left[\left(\frac{z_1}{z_2}\right)^{1/_2 p_n} - \left(\frac{z_1}{z_2}\right)^{1/_2 p_n}\right] = 0 \tag{5.40}$$

Since, according to the conditions of the problem, $z_1 \neq z_2$, then equation (5.40) becomes

$$\left(\frac{z_1}{z_2}\right)^{1/_2 p_n} - \left(\frac{z_1}{z_2}\right)^{-1/_2 p_n} = 0 \tag{5.41}$$

Equation (5.41) for p_n yields infinitely many imaginary roots, and these roots are determined by the formula

$$p_n = \frac{2m\pi i}{\ln (z_1/z_2)} \tag{5.42}$$

Here $\underline{m}$ can take on the value of any whole number ($m = 1, 2, 3, \ldots$).

It follows from formulas (5.38) and (5.42) that equation (5.36) will be satisfied only with $\mu < 0$ and $\mu > 1$, for all shells of negative curvature, for which the fundamental differential equation (5.36) is of the hyperbolic type. The particular unstable states of stress and bending deformation of such shells, explained by their negative Gaussian curvature, will occur for terms of the series (5.37) having $\underline{n}$ determined by the formula

$$\ln \frac{z_1}{z_2} = \frac{m\pi}{\sqrt{4\mu(\mu - 1)(n^2 - 1) - 1}}$$

Since $\underline{m}$ is any whole number, it follows from this that there exist certain relative dimensions z_1/z_2 for which a given shell can have infinitely many eigen-shapes of equilibrium and bending. Such shells will belong to a thin-walled geometrically variable three-dimensional system, as shown in the monograph [1]*. There are infinitely many degrees of geometric variability of these systems. As pointed out in the monograph [1]*, the design of such shells ought to proceed in

* The [1] appears to be a typographical error for [8] – Translator's note.

accordance with a bending theory. The bending and twisting moments arising from an arbitrarily assigned load in shells of negative curvature are not of a localized nature.

For shells of positive curvature, equation (5.36) is not satisfied for any real value of p_n. This means that the determinant of the equations, (5.40) cannot become zero with $0 < \mu < 1$. Hence, it follows that shells of positive curvature, in the presence at the edges $z = z_1$ and $z = z_2$ of "shear diaphragms", inextensible in their planes, are stable, statically determinate, three-dimensional systems.

Indeterminacy in the solution of the static problem for shells of positive curvature can arise only as a consequence of static indeterminacy in the constraining conditions at the edges of the shell, i.e., in case the shell, along an edge or part of an edge, is also constrained from vertical displacement, resisting a movement of the edge out of its plane.

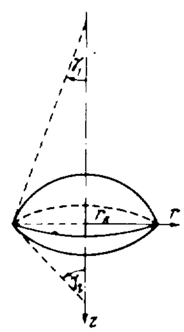


Fig. 7.

As has been repeatedly pointed out by the author, moments in shells of positive curvature have a regional, localized character. These moments can be determined in accordance with our general engineering theory of shallow shells.

6. Let us now examine a shell of revolution consisting of two spherical shells. Let the radii of these shells be a_1 and a_2. The radius of the parallel serving as the line of contact of the two shells is here designated by r_k. We designate by γ_1 and γ_2 the angles between the axis of revolution Oz and the tangents to the meridians of the shells at points along their line of contact. The positive sense of these angles is indicated on Fig. 7. The origin for the axial coordinate z for each of the two shells will be chosen in the geometric center of the appropriate shell.

Let us first examine the problem of equilibrium of the compound closed shell under the assumption that the angle $\varphi = \gamma_1 - \gamma_2$ between the tangents to the meridians of the two joined shells has a positive value. Writing the equilibrium conditions for an infinitesimally small element of the shell in the neighborhood of the line of contact, we obtain:

$$S_1 + \sin\gamma_1 \frac{\partial N_1}{\partial\beta} - \left(S_2 + \sin\gamma_2 \frac{\partial N_2}{\partial\beta}\right) = -q_\beta - \frac{\partial q_r}{\partial\beta} \qquad (5.43)$$
$$\cos\gamma_1 N_1 - \cos\gamma_2 N_2 = -q_z$$

Here S_1, N_1, S_2, and N_2 are the shear and normal stresses directed along the meridians for shells 1 and 2 respectively;

$$q_z = q_z(\beta), \qquad q_r = q_r(\beta), \qquad q_\beta = q_\beta(\beta)$$

are components of the external load per unit length, applied along the line of contact, and given as a function of the angular coordinate β. Of these components, q_z is the component acting parallel to the axis Oz, q_r is the component directed along the radius of the joining parallel, and q_β is the component directed along the tangent to this parallel.

The static boundary conditions necessary in the momentless theory, pertaining to points along the common line of contact of the two shells, are expressed by equations (5.43). We will assume that the compound shell examined here is acted on by forces applied only along the line of contact. These forces, represented in equations (5.43) by the linear loads q_z, q_r, and q_β, must be a system of forces in static equilibrium when the entire thin-walled system is considered. With these conditions, the harmonic functions for the two spherical shells must be selected such that the internal stresses determined by these functions have finite values everywhere, and go to zero at the poles of the shells — the lower pole being in shell 1 and the upper in shell 2.

Applying the series method as in the case of equations (5.31) and (5.36), and subjecting the harmonic functions to the above-formulated conditions of regularity at all points of the shells, we obtain

$$\varphi_1 = \sum A_n e^{-n\alpha_1} \cos n\beta, \quad \varphi_2 = \sum B_n e^{n\alpha_2} \cos n\beta \qquad (5.44)$$

Here α_1 is the independent variable for the lower shell (shell 1) and α_2 is the independent variable for the upper shell (shell 2); $\underline{n}$ is the number of the term in the appropriate series, taking on arbitrary positive integral values.

The harmonic function φ_1 for the first shell of radius a_1 is determined by the first of formulas (5.44). The independent variable α_1 in this formula can range between the limits

$$\operatorname{arcsh} \frac{z_{1k}}{\sqrt{a_1^2 - z_{1k}^2}} \leqslant \alpha_1 \leqslant \infty$$

Here z_{1k} is the axial coordinate of the line of contact, measured from the center O_1 of the first shell.

The harmonic function φ_2 for the second shell of radius a_2 is determined by the second of formulas (5.44). The independent variable α_2 in this formula ranges between the limits

$$-\infty \leqslant \alpha_2 \leqslant \operatorname{arc\,sh} \frac{z_{2k}}{\sqrt{a_2^2 - z_{2k}^2}}$$

Here z_{2k} is the axial coordinate of the line of contact, reckoned from the geometric center O_2 of the second shell.

For the shell shown in Fig. 7, with $\gamma_1 > 0$ and $\gamma_2 > 0$, both of the quantities z_{1k} and z_{2k} have negative values (the axis Oz is directed down).

We have the following formulas for the internal shear and normal meridional stresses of the two shells:

$$\begin{aligned} N_1 &= -\frac{a_1}{r_1^2} \sum A_n n e^{-n\alpha_1} \cos n\beta, & N_2 &= \frac{a_2}{r_2^2} \sum A_n n e^{n\alpha_2} \cos n\beta \\ S_1 &= \frac{a_1}{r_1^2} \sum B_n n e^{-n\alpha_1} \sin n\beta, & S_2 &= \frac{a_2}{r_2^2} \sum B_n n e^{n\alpha_2} \sin n\beta. \end{aligned} \tag{5.45}$$

The coefficients A_n and B_n ($n = 1, 2, 3, \ldots$) must be determined from the boundary conditions (5.43). Representing the right members of these equations by appropriate trigonometric series, we obtain a system of two simultaneous linear equations for the coefficients A_n and B_n associated with the nth term of the series (5.45).

$$\begin{gathered} -a_1 (1 - n \sin \gamma_1) e^{-n\alpha_{1k}} A_n + a_2 (1 + n \sin \gamma_2) e^{n\alpha_{2k}} B_n = -\frac{r_k^2}{n} P_n \\ -a_1 \cos \gamma_1 e^{-n\alpha_{1k}} A_n - a_2 \cos \gamma_2 e^{n\alpha_{2k}} B_n = -\frac{r_k^2}{n} Q_n \end{gathered} \tag{5.46}$$

Here a_1 and a_2 are the radii of the joined shells, r_k is the radius of the common line of joining, and P_n and Q_n are coefficients of the appropriate trigonometric series for the right members of equations (5.43). With the right members of equations (5.43) assigned as functions of β, these coefficients have completely determinate values.

The unknown coefficients A_n and B_n of the trigonometric series (5.45) will have completely determinate values if the determinant of equations (5.46) is different from zero for every positive integer $\underline{n}$. The general formula for this determinant has the form

$$\Delta_n = a_1 a_2 e^{n(\alpha_{2k} - \alpha_{1k})} [n \sin(\gamma_2 - \gamma_1) + \cos \gamma_2 + \cos \gamma_1]$$

It is clear from this formula that the compound shell being examined can be designed according to the momentless theory for arbitrary loads, subject to the condition that the quantity

$$C_n = n \sin(\gamma_2 - \gamma_1) + \cos\gamma_2 + \cos\gamma_1 \tag{5.47}$$

does not become zero for any positive integral value of n. This will be the case when the angle of contiguity $\varphi = \gamma_2 - \gamma_1$, between the tangents to the arcs of the two circles at their point of contact does not have a negative value.

If this angle of contiguity $\varphi = \gamma_2 - \gamma_1$ at the point of contact of the two circles has a negative value (Fig. 8), then the determinant of the system of equations (5.46) can become zero. These particular unstable solutions will occur if the right member of the equation

$$n = -\frac{\cos\gamma_1 + \cos\gamma_2}{\sin(\gamma_2 - \gamma_1)} \tag{5.48}$$

is a positive integer.

As an appendix which provides great insight into the problem considered here, we shall assume that the junction of one shell with the other is accomplished with the help of a third shell described by part of the surface of a torus with arbitrarily small radius. In other words, we assume that meridians of the two joined spherical shells are smoothly connected without corner points, by means of circular arcs of arbitrarily small radius. In such a geometric interpretation, the angle of contiguity $\varphi = \gamma_2 - \gamma_1$, together with the curvature of the parallel at the junction, determine the local value of the Gaussian curvature of the surface, referred to an elementary strip at the transition from one shell to the other.

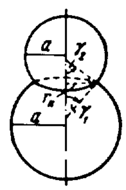


Fig. 8.

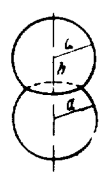


Fig. 9.

It follows from formula (5.47) that the shell shown in Fig. 7, and characterized by the fact that the Gaussian curvature in the neighborhood of the junction has a positive value $\gamma_2 - \gamma_1 > 0$, is a rigid, thin-walled system, not permitting any bending in the absence of stretching. The shell shown in Fig. 8, having negative curvature $\gamma_2 - \gamma_1 < 0$ in the neighborhood of the junction, with the angles γ_1 and γ_2 giving positive integral values for n in the formula (5.48), is, according to the momentless theory, an unstable, geometrically variable system, permitting infinitesimally small bending.

If the trigonometric quantities in formula (5.48) are expressed through the basic dimensions of the joined shells, then we have

$$n = \frac{a_1 + a_2}{h} \tag{5.49}$$

Here a_1 and a_2 are radii of the shells, h is the distance along the axis of revolution between their centers O_1 and O_2. In accordance with the above exposition, formulas (5.48) and (5.49) properly refer only to those shells for which the local curvature at an arbitrarily small elementary strip of the junction has a negative value.

For the symmetrical shell shown in Fig. 9, we have

$$n = \frac{2a}{h} = \frac{d}{h}$$

It follows from this formula that the examined symmetrical shell with negative curvature in the zone of the junction will have particular, unstable solutions in all those cases for which the distance h between the centers of the shells is contained an integral number of times in the diameter d.

In the special case in which $h = \frac{1}{2}d$, we obtain $n = 2$. This means that for the given shell, unstable shapes of equilibrium and bending will be associated only with the second term of the appropriate trigonometric series. In such a shell, infinite stresses arise from a momentless self-equilibrating load corresponding to the second term of the series, distributed along the parallel according to the law $\cos 2\beta$.

One should note that the shells examined here, having negative curvature only along certain lines, in contrast to shells of the hyperbolic type with negative curvature over the surface, can, with given dimensions, have only one degree of freedom of geometric variability, and not an infinite number.

§ 6. Conical elastic shells. 1. The middle surface of a conical shell is referred to the coordinates z and β.

The angle β will be considered positive when it is clockwise looking at the shell along the positive z axis (from the top down in Fig. 10).

We have the following formula for the radius of a parallel of the shell:

$$r = r_1 + z \operatorname{tg} \gamma \tag{6.1}$$

Here r_1 is the radius of the parallel at $z = 0$, and γ is the angle between the z azis and the generators of the cone. This angle will be considered positive if the radius r of a parallel increases with an increase in the coordinate z.

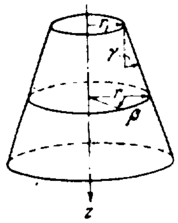


Fig. 10.

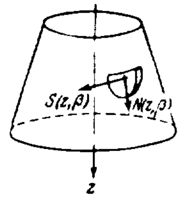


Fig. 11.

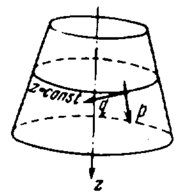


Fig. 12.

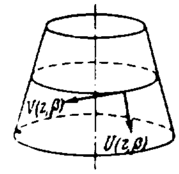


Fig. 13.

In the sequel, we shall use the following abbreviated notation for trigonometric functions of the angle γ.

$$s = \sin\gamma, \qquad c = \cos\gamma, \qquad t = \operatorname{tg}\gamma \tag{6.2}$$

The theory of elastic equilibrium of a momentless conical shell is described by the differential equations

$$\frac{\partial}{\partial z}(rN) + \frac{1}{c}\frac{\partial S}{\partial \beta} + p = 0, \qquad \frac{\partial}{\partial z}(rS) + tS + q = 0 \tag{6.3}$$

$$\frac{\partial u}{\partial z} = \frac{N}{Ehc}, \qquad \frac{1}{r}\frac{\partial u}{\partial \beta} + cr\frac{\partial}{\partial z}\left(\frac{v}{r}\right) = \frac{2(1+\nu)}{Eh}S \tag{6.4}$$

The conditions of tangential equilibrium of an element of the shell are expressed by the first two equations. In these equations, $S = S(z, \beta)$ is the shear stress; $N = N(z, \beta)$ is the normal stress associated with the stretching of the shell along the generators (Fig. 11); $\underline{p}$ and $\underline{q}$ are components of the surface load directed along a generator and along the tangent to a parallel, respectively (Fig. 12).

The relations between the stresses and the strains for the elastic shell are expressed by the third and fourth equations, in which the strains for a particular shape are determined by the derivatives of the displacements, and the stresses by the internal forces. In these equations, $u = u(z, \beta)$ and $v = v(z, \beta)$ are the tangential components of the total displacement vector of a point, directed along a generator and along the tangent to a parallel, respectively (Fig. 13); E is the modulus of elasticity; $\underline{z}$ is Poisson's ratio; and $\underline{h}$ is the thickness of the shell.

The normal circumferential stress is absent in equations (6.3) and (6.4). For a conical shell this stress is easily determined as a quantity proportional to the normal component of the surface load. If this component is equal to zero, then the circumferential stress will also be equal to zero.

In the sequel, we shall examine the problem resulting from equations (6.3) when the surface load is absent. We will have a system of four homogeneous equations in four unknown functions.

In order to obtain a general integral of equations (6.3) and (6.4), we will make use of the method of initial functions which we proposed in the theory of shells. According to this method, the tangential stresses N_1 and S_1 and the tangential displacements u_1 and v_1 along an initial parallel $z = 0$, considered as functions of the angular coordinate β along this parallel, are taken as fundamental factors determining the stressed and strained state of the momentless shell. The positive directions of the stresses N_i, S_i and the displacements u_i, v_i along an initial parallel $z = 0$, and along any other, z = const., are shown in Fig. 14.

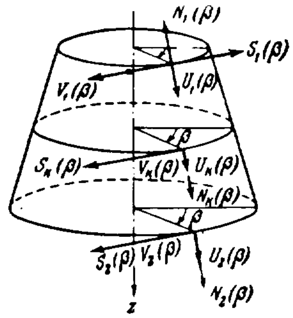


Fig. 14.

With $p = q = 0$, we will have these general formulas for the unknown functions of equations (6.3, 6.4):

$$S_k = \left(\frac{r_1}{r}\right)^2 S_1(\beta) \tag{6.5}$$

$$N_k = \frac{r_1}{r_2} N_1 - \frac{1}{s}\frac{r_1}{r}\left(1 - \frac{r_1}{r}\right) S_1'(\beta)$$

$$Ehu_k = Ehu_1 + \frac{r_1}{s}\ln\left(\frac{r}{r_1}\right) N_1(\beta) + \frac{r_1^2}{s^2}\left(1 - \frac{r_1}{r} - \ln\frac{r}{r_1}\right) S_1'(\beta)$$

$$Ehv_k = \frac{r}{r_1} Ehv_1 + \frac{1}{s}\left(1 - \frac{r}{r_1}\right) Ehu_1' + \frac{r_1}{s^2}\left(1 - \frac{r}{r_1} + \ln\frac{r}{r_1}\right) N_1'(\beta) +$$
$$+ \frac{r_1}{s^3}\left(r\frac{1}{2r_1} - \frac{r_1}{2r} - \ln\frac{r}{r_1}\right) S_1''(\beta) + \frac{1+\nu}{s} r_1\left(\frac{r}{r_1} - \frac{r_1}{r}\right) S_1(\beta)$$

In the left members of these formulas stand the unknown quantities S_k, N_k, Ehu_k, Ehv_k, which pertain to any parallel z = const. (see Fig. 14), and which are functions of the two coordinates $\underline{z}$ and β.

The index $\underline{k}$ indicates that the quantities determined by formulas (6.5) pertain to points along a parallel with running coordinate $\underline{z}$. The radius of this parallel is denoted by $\underline{r}$.

In the right members of equations (6.5) stand quantities pertaining to the initial functions S_1, N_1, Ehu_1, Ehv_1 and their derivatives:

$$S_1' = \frac{dS_1}{d\beta}, \quad S_1'' = \frac{d^2S_1}{d\beta^2}, \quad N_1' = \frac{dN_1}{d\beta}, \quad u_1' = \frac{du_1}{d\beta}$$

As quantities determined for points of the initial (fixed) parallel $z = 0$ (the radius of this parallel is denoted by r_1), these functions and their derivatives depend only on the single angular coordinate β. With $r = r_1$, the formulas (6.5)

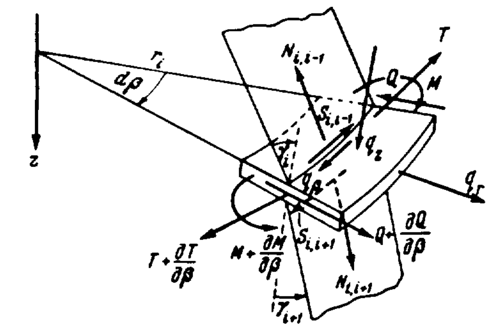


Fig. 16.

$$w_i = -\frac{\partial v_i}{\partial \beta} \tag{6.10}$$

Making use of the formula thus obtained, we record the value of the bending moment acting in the ring, expressed in terms of the tangential displacement:

$$M_i = -\frac{EI_i}{r_i^2}\left(\frac{\partial^3 v_i}{\partial \beta^3} + \frac{\partial v_i}{\partial \beta}\right) \tag{6.11}$$

Further, taking advantage of the second of the general formulas (6.6), we express the derivatives of the shear forces, transmitted by the shell to the ring in terms of the normal forces:

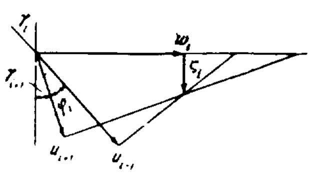
$$\begin{aligned} S'_{i,i+1} &= -N_{i+1,i}\frac{s_{i+1}r_{i+1}^2}{r_i(r_{i+1}-r_i)} + \frac{s_{i+1}r_{i+1}}{r_{i+1}-r_i}N_{i,i+1} \\ S'_{i,i-1} &= -N_{i,i-1}\frac{s_i r_{i-1}}{r_i - r_{i-1}} + N_{i-1,i}\frac{s_i r_{i-1}^2}{r_i(r_i - r_{i-1})} \end{aligned} \tag{6.12}$$

Introducing the value obtained for the bending moment (6.11) and the shear forces (6.12) into the second equation (6.9), we obtain:

$$N_{i,i+1}c_{i+1} - N_{i,i-1}c_{i-1} + q_{iz} = 0$$

$$\begin{aligned} &\frac{EI}{r_i^4}\left(\frac{\partial^2}{\partial\beta^2}+1\right)^2\frac{\partial^2 v_i}{\partial \beta^2} + s_{i+1}\frac{\partial^2 N_{i,i+1}}{\partial\beta^2} - s_i\frac{\partial^2 N_{i,i-1}}{\partial\beta^2} + \\ &+ \frac{s_{i+1}}{(r_i - r_{i+1})}\frac{r_{i+1}^2}{r_i}N_{i+1,i} - \frac{s_{i+1}r_{i+1}}{r_i - r_{i+1}}N_{i,i+1} - \frac{s_i r_{i-1}^2}{(r_i - r_{i-1})r_i}N_{i-1,i} + \\ &+ \frac{s_i r_{i-1}}{r_i - r_{i-1}}N_{i,i-1} + \frac{\partial^2 q_{ir}}{\partial\beta^2} + \frac{\partial q_{i\beta}}{\partial\beta} = 0 \end{aligned} \tag{6.13}$$

In addition to the equations of equilibrium, the conditions of continuity of the strain must be fulfilled in the neighborhood of each ring. The latter conditions are obtained by equating the displacements of the ring and the two adjoining conical shells at their place of joining to the ring.



Using Fig. 17, we can write:

$$w_i = \frac{1}{\sin\varphi_i}(u_{i-} \cos\gamma_{i+1} - u_{i+1}\cos\gamma_i)$$

Differentiating this formula once with respect to β, replacing the radial displacement by the tangential in accordance with formula (6.10), and introducing the previous abbreviated notation for the conical angles we finally obtain the

Here the determined quantities standing in the left members of the equations pertain to arbitrary points of a parallel with running coordinate z; quantities designated by the index 1 are the initial functions and their derivatives, which pertain to points of the parallel $z = z_1$, and depend only on the single coordinate β.

2. Let us examine a system of conical shells, stiffened at the junctions by rings (Fig. 15).

As before, we refer the middle surface of the shell to the cylindrical system of coordinates z and β. In order to obtain the differential equations of equilibrium, taking into account the effect of the ring on the two neighboring momentless shells, we separate an element in the neighborhood of one of the rings by the lines β = const. and b + $d\beta$ = const. and we replace the discarded parts by forces (Fig. 16).

The equilibrium conditions of the separated element of the ring take the form

$$\begin{aligned}
&\sum(z) = 0, && N_{i,\,i+1}c_{i+1} - N_{i,\,i-1}c_{i-1} + q_{iz} = 0 \\
&\sum(\beta) = 0, && \frac{\partial T_i}{\partial\beta} + Q_i + r_i(S_{i,\,i+1} - S_{i,\,i-1}) + r_i q_{i\beta} = 0 \\
&\sum(r) = 0, && \frac{\partial Q_i}{\partial\beta} - T_i + r_i(N_{i,\,i+1}s_{i+1} - N_{i,\,i-1}s_i) + r_i q_{ir} = 0 \\
&\sum(M_i) = 0, && \frac{\partial M_i}{r_i\partial\beta} + Q_i = 0
\end{aligned} \tag{6.8}$$

Eliminating the forces Q_i and T_i from the obtained system,

$$\begin{gathered}
N_{i,\,i+1}c_{i+1} - N_{i,\,i-1}c_{i-1} + q_{iz} = 0 \\
-\frac{1}{r_i^2}\left(\frac{\partial^2}{\partial\beta^2} + 1\right)\frac{\partial^2 M_i}{\partial\beta^2} + s_{i+1}\frac{\partial^2 N_{i,\,i+1}}{\partial\beta^2} - s_i\frac{\partial^2 N_{i,\,i-1}}{\partial\beta^2} + \\
+ \frac{\partial S_{i,\,i-1}}{\partial\beta} - \frac{\partial S_{i,\,i-1}}{\partial\beta} + \frac{\partial^2 q_{iz}}{\partial\beta^2} + \frac{\partial q_{i\beta}}{\partial\beta} = 0
\end{gathered} \tag{6.9}$$

Let us express the bending moment arising in the ring in terms of the radial displacement:

$$M_i = \frac{EI}{r_i^2}\left(\frac{\partial^2 w_i}{\partial\beta^2} + w_i\right)$$

If we neglect the deviation of the tangential strain from its original tangential direction along the ring's line of contact with the shell, then the radial displacement w can be expressed in terms of the derivative of the tangential displacement according to the formula become an identity, showing that the quantities $S_1 = S_1(\beta)$, $N_1 = N_1(\beta)$, $u_1 = u_1(\beta)$, $v_1 = v_1(\beta)$, playing the role of arbitrary functions of β in the integration of the basic equations (6.3, 6.4) with $p = q = 0$, are the initial functions. The stressed and strained state of a conical elastic shell is determined in a unique manner by these functions and their derivatives, as we see. In particular, setting $r = r_2$ in formulas (6.5), we obtain values of the stresses and displacements for points along the lower parallel of fixed radius r_2 (see Fig. 14):

$$
\begin{aligned}
S_2 &= \left(\frac{r_1}{r_2}\right)^2 S_1(\beta), \qquad N_2 = \frac{r_1}{r_2} N_1 - \frac{1}{s}\frac{r_1}{r_2}\left(1 - \frac{r_1}{r^2}\right) S_1'(\beta) \\
Ehu_2 &= Ehu_1 + \frac{r_1}{s} \ln\left(\frac{r_2}{r_1}\right) N_1(\beta) + \frac{r_1^2}{s^2}\left(1 - \frac{r_1}{r_2} - \ln\frac{r_2}{r_1}\right) S_1'(\beta) \\
Ehv_2 &= \frac{r_2}{r_1} Ehv_1 + \frac{1}{s}\left(1 - \frac{r_2}{r_1}\right) Ehu_1' + \frac{r_1}{s^2}\left(1 - \frac{r_2}{r_1} + \ln\frac{r_2}{r_1}\right) N_1'(\beta) + \\
&+ \frac{r_1}{s^3}\left(\frac{r_2}{2r_1} - \frac{r_1}{2r_2} - \ln\frac{r_2}{r_1}\right) S_1''(\beta) + \frac{1+\nu}{s} r_1 \left(\frac{r_2}{r_1} - \frac{r_1}{r_2}\right) S_1(\beta)
\end{aligned}
\tag{6.6}
$$

If the stresses $N_2(\beta)$, $S_2(\beta)$ and the displacements $u_2(\beta)$, $v_2(\beta)$, pertaining to the lower parallel r_2 are taken as the initial functions of β, then formulas for the stresses $N_1(\beta)$, $S_1(\beta)$ and the displacements $u_1(\beta)$, $v_1(\beta)$ at points along the upper parallel will be obtained from formulas (6.6) by interchanging the indices, 1 by 2 and 2 by 1.

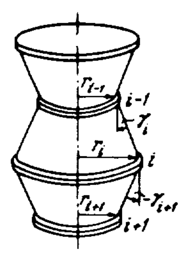


Fig. 15.

By means of formulas (6.6) there is established, in this way, a well-defined mutual correspondence between the tangential stresses and displacements along any two parallels of the shell.

Setting $r = r_1 + tz$ in formulas (6.5), and then passing to the limit as $\gamma \to 0$, we clearly obtain also, by the method of initial functions, general formulas pertaining to a cylindrical shell with radius $r = \text{const}$.

These formulas will have the following form:

$$
\begin{aligned}
S_2 &= S_1, \qquad N_2 = N_1 = \frac{z - z_1}{r} S_1'(\beta \\
Ehu_2 &= Ehu_1 + (z - z_1) N_1(\beta) + \frac{(z - z_1)^2}{2r} S_1'(\beta \\
Ehv_2 &= Ehv_1 - \frac{(z - z_1)}{r} Ehu_1' + \frac{(z - z_1)^2}{2r} N_1'(\beta) + \\
&+ \frac{(z - z_1)^3}{6r^2} S_1''(\beta) + 2(1+\nu)(z + z_1) S_1(\beta)
\end{aligned}
\tag{6.7}
$$

equation of continuity of the strain in the following form:

$$c_{i+1}u'_{i-1} - c_i u'_{i+1} + \dot{v}_i \sin\varphi_i = 0 \tag{6.14}$$

Further, using the fourth of the general formulas (6.6) from the method of initial functions, we express the longitudinal displacements in terms of the tangential displacements and the normal forces, and substitute into (6.14):

$$\begin{aligned}
&\frac{c_{i+1}s_i}{r_i-r_{i-1}} r_i v_{i-1} - \left(\frac{c_{i+1}s_i r_{i-1}}{r_i-r_{i-1}} - \frac{c_i s_{i+1} r_{i+1}}{r_i - r_{i+1}}\right) v_i + \dot{v}_i \sin\varphi_i - \\
&- \frac{c_i s_{i+1} r_i}{r_i - r_{i+1}} v_{i+1} + \frac{r_{i-1}c_{i+1}}{(r_i-r_{i-1})s_i}\left(\frac{r_i}{2}+\frac{r_{i-1}}{2} - \frac{r_i r_{i-1}}{r_i-r_{i-1}}\ln\frac{r_i}{r_{i-1}}\right)\frac{N'_{i-1,i}}{Eh_i} - \\
&-\frac{r_i c_{i+1}}{(r_i-r_{i-1})s_i}\left(\frac{3r_i}{2}-\frac{r_{i-1}}{2}-\frac{r_i^2}{r_i-r_{i-1}}\ln\frac{r_i}{r_{i-1}}\right)\frac{N'_{i,i-1}}{Eh_i}+ \\
&+\frac{c_i r_i}{(r_i-r_{i+1})s_{i+1}}\left(\frac{3r_i}{2}-\frac{r_{i+1}}{2}-\frac{r_i^2}{r_i-r_{i+1}}\ln\frac{r_i}{r_{i+1}}\right)\frac{N'_{i,i+1}}{Eh_{i+1}}- \\
&-\frac{r_{i+1}c_i}{(r_i-r_{i+1})s_i}\left(\frac{r_i}{2}+\frac{r_{i+1}}{2}-\frac{r_i r_{i+1}}{r_i-r_{i+1}}\ln\frac{r_i}{r_{i+1}}\right)\frac{N_{i+1,i}}{Eh_{i+1}}+ \\
&+\frac{(1+\nu)(r_i+r_{i-1})\,c_{i+1}s_i}{(r_i-r_{i-1})Eh_i}\int_\beta (r_{i-1}N_{i-1,i}-r_iN_{i,i-1})\,d\beta - \\
&-\frac{(1+\nu)(r_i+r_{i+1})\,c_i s_{i+1}}{(r_i-r_{i+1})Eh_{i+1}}\int_\beta(r_{i+1}N_{i+1,i}-r_iN_{i,i+1})\,d\beta = 0
\end{aligned} \tag{6.15}$$

In order to solve the equations obtained, we introduce into the investigation, functions N_i, defined by the formulas:

$$N_{i-1,i} = c_{i-1}\frac{N_{i-1}'}{r_{i-1}}, \quad N_{i,i+1} = c_i\frac{N_i'}{r_i}$$

$$N_{i,i-1} = c_{i+1}\frac{N_i'}{r_i}+\frac{1}{c_i}q_{iz}, \quad N_{i+1,i} = c_{i+2}\frac{N_{i+1}'}{r_{i+1}}+\frac{1}{c_{i+1}}q_{i+1,z}$$

Then, denoting the coefficients of the unknown functions

$$a_{ii} = -\sin\varphi_i, \qquad b_{i,i-1} = -\frac{s_i c_{i-1} r_{i-1}}{r_i-r_{i-1}}$$

$$b_{i,i} = \frac{s_i c_{i+1} r_{i-1}}{r_i-r_{i-1}}-\frac{s_{i+1}c_i r_{i+1}}{r_i-r_{i+1}}, \qquad b_{i,i+1} = \frac{s_{i+1}c_{i+2}r_{i+1}}{r_i-r_{i+1}}$$

$$s_{i,i-1} = \frac{c_{i-1}c_{i+1}}{(r_i-r_{i-1})s_iEh_i}\left(\frac{r_i+r_{i-1}}{2}-\frac{r_ir_{i-1}}{r_i-r_{i-1}}\ln\frac{r_i}{r_{i-1}}\right)$$

$$\begin{aligned}
s_{i,i} = {}&\frac{c_{i-1}c_{i+1}}{(r_i-r_{i-1})s_iEh_i}\left(\frac{3r_i}{2}-\frac{r_{i+1}}{2}-\frac{r_i^2}{r_i-r_{i+1}}\ln\frac{r_i}{r_{i+1}}\right) - \\
&-\frac{c_{i+1}^2}{(r_i-r_{i-1})s_iEh_i}\left(\frac{3r_i}{2}-\frac{r_{i-1}}{2}-\frac{r_i^2}{r_i-r_{i-1}}\ln\frac{r_i}{r_{i-1}}\right)
\end{aligned}$$

$$s_{i,i+1} = -\frac{c_ic_{i+2}}{(r_i-r_{i+1})Eh_{i+1}s_{i+1}}\left(\frac{r_i}{2}+\frac{r_{i+1}}{2}-\frac{r_ir_{i+1}}{r_i-r_{i+1}}\ln\frac{r_i}{r_{i+1}}\right)$$

$$t_{i,\,i-1} = \frac{(1+\nu)(r_i + r_{i-1})\, c_{i+1}\, c_{i-1}\, s_i}{(r_i - r_{i-1})\, Eh_i}$$

$$t_{i,\,i} = -\frac{(1+\nu)(r_i + r_{i-1})\, c_{i+1}^2\, s_i}{(r_i - r_{i-1})\, Eh_i} + \frac{(1+\nu)(r_i + r_{i+1})\, c_i^2\, s_{i+1}}{(r_i - r_{i+1})\, Eh_{i+1}},$$

$$t_{i,\,i+1} = -\frac{(1+\nu)(r_i + r_{i+1})\, c_i\, c_{i+2}\, s_{i+1}}{(r_i - r_{i-1})\, Eh_{i+1}}$$

$$P_i = -\frac{r_i s_i}{c_i} q_{zi}'' + \frac{r_i r_{i-1} s_i}{(r_i - r_{i-1})\, c_i} q_{zi} + \frac{r_{i+1}^2 s_{i+1}}{(r_i - r_{i+1})\, c_{i+1}} q_{z,\,i+1} + r_i (q_{zi}^{\cdot} + q_{\beta i}')$$

$$Q_i = -\frac{r_i c_{i+1}}{(r_i - r_{i-1})\, c_i s_i}\left(\frac{3r_i}{2} - \frac{r_{i-1}}{2} - \frac{r_i^2}{r_i - r_{i-1}} \ln \frac{r_i}{r_{i-1}}\right)\frac{q_{zi}'}{Eh_i} -$$

$$-\frac{r_{i+1} c_i}{(r_i - r_{i+1})\, c_{i+1} s_{i+1}}\left(\frac{r_i}{2} + \frac{r_{i+1}}{2} - \frac{r_i r_{i+1}}{r_i - r_{i+1}} \ln \frac{r_i}{r_{i+1}}\right)\frac{q_{z,\,i+1}'}{Eh_{i+1}} -$$

$$-\frac{(1+\nu)(r_i + r_{i-1})}{r_i - r_{i-1}} \frac{c_{i+1} s_i r_i}{c_i} \int\limits_{\beta} \frac{q_{zi}}{Eh_i} d\beta - \frac{(1+\nu)(r_i + r_{i+1})}{r_i - r_{i+1}} \frac{c_i s_{i+1} r_{i+1}}{c_{i+1}} \int\limits_{\beta} \frac{q_{z,\,i+1}}{Eh_{i+1}} d\beta$$

we obtain a system of two equations of the following form for each junction of the shells:

$$\begin{gathered} \frac{EI_i}{r_i^3}\left(\frac{d^2}{d\beta^2} + 1\right)^2 v_i''' + a_{ii} N_i''' + \sum_{k=i-1}^{i+1} b_{ik} N_k' + P_i = 0 \\ -a_{ii} v_i'' - \sum_{k=i-1}^{i+1} b_{ik} v_k + \sum_{k=i-1}^{i+1} S_{ik} N_k'' + \sum_{k=i-1}^{i+1} t_{ik} N_k + Q_i = 0 \end{gathered} \qquad (6.16)$$

If the shell has a structure which is inextensible in its tangent plane, then the quantities Q_i, s_{in}, and t_{in} reduce to zero, and equa ions (6.16) are simplified:

$$\frac{EI_i}{r_i^3}\left(\frac{d^2}{d\beta^2} + 1\right)^2 v_i''' + a_{ii} N_i''' + \sum_{k=i-1}^{i+1} b_{ik} N_k' + P_i = 0, \quad -a_{ii} v_i''' - \sum_{k=i-1}^{i+1} b_{ik} v_k = 0 \qquad (6.17)$$

In the absence of stiffening ribs at the joints of the shells, the system of equations (6.17) breaks up into two independent systems:

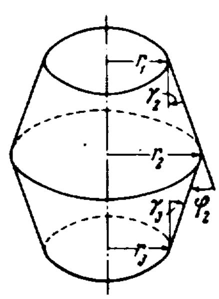


Fig. 18.

$$a_{ii} N_i''' + \sum_{k=i-1}^{i+1} b_{ik} N_k' + P_i = 0 \qquad (6.18)$$

$$a_{ii} v_i'' - \sum_{k=i-1}^{i+1} b_{ik} v_k = 0 \qquad (6.19)$$

Equations 6.18) and (6.19) have a similar structure, which is a consequence of the static-geometric analogy.

By way of an example, let us examine two conical shells (Fig. 18), not having a reinforcing ring at the junction, and constrained at the ends by inextensible "shear diaphragms":

$$r_1 = r_3, \quad \gamma_2 = \gamma_3, \quad EI_2 = 0, \quad N_1 = N_3 = 0, \quad v_1 = v_3 = 0$$

$$a_{22} = -\sin\varphi_2, \quad b_{22} = \frac{\sin\gamma_2 \cos\gamma_3 r_1}{r_2 - r_1} + \frac{\sin\gamma_3 \cos\gamma_2 r_3}{r_2 - r_3} = \frac{r_1}{r_2 - r_1}\sin\varphi_2$$

Substituting the coefficients obtained into equations (6.18, 6.19), we obtain two identical equations for the static function N_2 and the geometric function v_2 respectively, which appears as a consequence of the static-geometric analogy:

$$(N_2')''\sin\varphi_2 - \frac{r_1}{r_2 - r_1}(N_2')\sin\varphi_2 = 0, \qquad v_2''\sin\varphi_2 - \frac{r_1}{r_2 - r_1}v_2\sin\varphi_2 = 0$$

or cancelling $\sin\varphi_2$, we obtain

$$(N_2')'' - \frac{r_1}{r_2 - r_1}(N_2') = 0, \qquad v_2'' - \frac{r_1}{r_2 - r_1}v_2 = 0 \qquad (6.20)$$

With $r_2 > r_1$, the integral of either of equations (6.20) has the form:

$$v_2 = C_1 \sin n\beta$$

where n is determined by the relation

$$n^2 + \frac{r_1}{r_2 - r_1} = 0 \quad \text{or} \quad r_2 = \frac{n^2 - 1}{n^2} r_1$$

The radius ratios for which n takes on integral values will correspond to different shapes of geometric variability of the shell.

Thus, with $r_2 = 3r_1/4$, $n = 2$, and the shape of the geometric variability is characterized by bending deformations of the shells along their line of contact, following the law $v_2 = C_1 \sin 2\beta$, where C_1 is an arbitrary constant.

LITERATURE CITED

[1] V.Z. Vlasov, "Design of shells of revolution with a non-symmetrical load," Proekt i Standart, Nos. 3 and 4, (1937).

[2] V.Z. Vlasov, "Design of shells described by surfaces of second order." Sb. TS N I I P S, Plates and Shells, State Construction Press, (1939).

[3] A.L. Gol'denveizer, A.K. Mroshchinskii and G.V. Repman, "Methods of design of spherical domes according to the momentless theory." Sb. TS N I I P S, Plates and Shells, (1939.

[4] V.V. Sokolovskii, "Equations of equilibrium of momentless shells." Prikladnaia Matematika i Mekhanika VII, No. 1, (1943).

[5] Iu. N. Rabotnov, "Certain solutions in the momentless theory of shells." Prikladnaia Matematika i Mekhanika X, Nos. 5-6, (1949).

[6] V.Z. Vlasov, "Momentless theory of thin shells, described by surfaces of revolution." Prikladnaia Matematika i Mekhanika XI, No. 4, (1947).

[7] V.V. Novoshilov, Theory of Thin Shells. Gostenkhizdat, (1947).

[8] V.Z. Vlasov, General Theory of Shells. Gostekhizdat, (1949).

[9] A.L. Gol'denveizer, Thin Elastic Shells. Gostekhizdat, (1952).

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